Viscous flow through a grating or lattice of cylinders

By JOSEPH B. KELLER
Courant Institute of Mathematical Sciences,
New York University, New York, N.Y.

(Received 15 July 1963 and in revised form 3 September 1963)

Viscous flow perpendicular to a line (or 'grating') of evenly spaced identical cylinders is considered in the case when the spacing between the cylinders is much smaller than their cross-sectional dimensions. Lubrication theory is used to find the pressure drop across the grating and hence the force on each cylinder. A square array (or 'lattice') of closely packed cylinders is similarly treated.

Let us consider the slow two-dimensional flow of a viscous fluid transverse to a grating or a lattice of identical cylinders (e.g. pipes or rods). This flow has been analysed for cylinders of special shape when the separation between any two of them is large compared to their diameter (Hasimoto 1959 and references therein). We wish to point out that in the opposite extreme of closely packed cylinders the flow can also be analysed, and rather simply. This is possible because then most of the flow resistance is concentrated in the narrow gaps between adjacent cylinders. The flow in these gaps can be determined in the manner customarily employed in the theory of lubrication. Therefore the flow resistance, the pressure gradient and the force on a rod can all be calculated. We have previously used similar considerations to find the effective electrical conductivity of a composite medium composed of a dense array of perfectly conducting spheres or cylinders in a medium of finite conductivity (Keller 1963; Buchal & Keller 1963).

In figure 1 is shown the gap between two adjacent cylinders assumed to be symmetric about the x-axis. This axis is in the flow direction and is midway between the cylinders, the equations of which are $y = \pm h(x)$. Assuming the flow to be locally parallel for each value of x, we see that the x-component of velocity is $u(x) = (y^2-h^2)\,p_x/2\mu$. Here p is the pressure, assumed to depend upon x only, and μ is the viscosity. The total volume flux is $q = \int_{-h}^{h} u\,dy = -2h^3p_x/3\mu$. Thus $p_x = -3\mu q/2h^3(x)$ and

$$p(x_1)-p(x_2) = \frac{3\mu q}{2}\int_{x_1}^{x_2}\frac{dx}{h^3(x)}. \tag{1}$$

If x_1 and x_2 are at opposite ends of the gap, (1) gives the pressure drop across the gap in terms of the flux q through it. When the two cylinders in figure 1 are part of a grating lying along the y-axis, (1) gives the pressure drop across the grating. The flux per unit length through the grating, which is also the average flow speed, is $U = Nq$ where N is the number of gaps per unit length. The force

F on a cylinder is just the pressure drop divided by the number of cylinders per unit length, so that

$$F = [p(x_1) - p(x_2)]/N = \frac{3\mu U}{2N^2}\int_{x_1}^{x_2} \frac{dx}{h^3(x)}. \tag{2}$$

When there is a lattice of cylinders consisting of M parallel gratings per unit length along the x-axis, the pressure drop per unit distance along the x-axis is M times the value given by (1). The force on a cylinder is still given by (2).

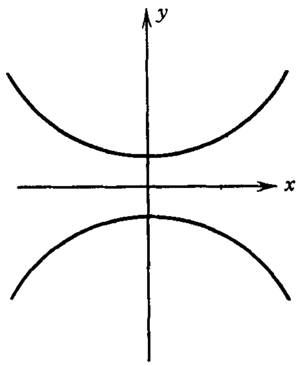

FIGURE 1. The gap between two neighbouring cylinders. The x-axis is parallel to the direction of flow and midway between the cylinders, which are symmetric about it. The equations of the cylinder are $y = \pm h(x)$.

To exemplify the above considerations, let us consider a grating of circular cylinders each of radius a with axes a distance $2c$ apart. Then

$$h(x) = c - (a^2 - x^2)^{\frac{1}{2}}$$

and

$$p(-a) - p(a) = 3\mu q \int_0^a [c - (a^2 - x^2)^{\frac{1}{2}}]^{-3}\, dx \approx \frac{9\pi\mu q a^{\frac{1}{2}}}{8\,.\,2^{\frac{1}{2}}(c-a)^{\frac{5}{2}}}. \tag{3}$$

In (3) we have retained only the term which is most important when $c-a$ is small. Now $N = 1/2c$ so (2) and (3) yield for the force on a cylinder, recalling that $a \approx c$,

$$F = \frac{9\pi\mu q c^{\frac{3}{2}}}{4\,.\,2^{\frac{1}{2}}(c-a)^{\frac{5}{2}}} = \frac{9\pi\mu U}{2\,.\,2^{\frac{1}{2}}}\left(1 - \frac{a}{c}\right)^{-\frac{5}{2}}. \tag{4}$$

If the cylinders are arranged in a square lattice then $M = N = 1/2c$ and the average pressure gradient $\overline{p}_x$ along the x-axis is M times (3), which yields

$$\overline{p}_x = \frac{9\pi\mu U c^{\frac{1}{2}}}{8\,.\,2^{\frac{1}{2}}(c-a)^{\frac{5}{2}}}. \tag{5}$$

Naturally (4) and (5) yield $\overline{p}_x = MNF = F/4c^2$ as must be the case. We see from (4) and (5) that for fixed values of μ, U and a, both $\overline{p}_x$ and F become infinite as $c-a$ tends to zero. It was to determine this singular behaviour that the present method was designed.

For a square lattice of circular cylinders with $a/c \ll 1$, Hasimoto (1959) has obtained the following result, given by his equation (6.5)

$$F \doteq 4\pi\mu U[\log(2c/a) - 1{\cdot}3105]^{-1}. \tag{6}$$

When $a/c \approx 1$, F is given instead by our equation (4). Thus (4) and (6) determine $F(a/c)$ near the two ends of the range $0 \leqslant a/c \leqslant 1$. For intermediate values of a/c, F can be found by numerical solution of an appropriate boundary value problem for the stream function of the flow.

The research reported in this paper was supported by the NSF under Grant No. GP 2003, NSF-22.

REFERENCES

BUCHAL, R. N. & KELLER, J. B. 1963 *J. Appl. Phys.* **34**, 3414.
HASIMOTO, H. 1959 *J. Fluid Mech.* **5**, 317.
KELLER, J. B. 1963 *J. Appl. Phys.* **34**, 991.